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FIXED POINT OF MULTIVALUED CONTRACTIONS IN ORTHOGONAL MODULAR METRIC SPACES

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Abstract. In this paper we generalize the notion of $O-$set and establish some fixed point theorems for $\perp - \alpha - \psi-$contraction multifunction in the setting of orthogonal modular metric spaces. As consequences of these results we deduce some theorems in orthogonal modular metric spaces endowed with a graph and partial order. Finally, we establish some theorems for integral type contraction multifunctions and give some examples to demonstrate the validity of the results.

Keywords. Fixed point theorem; metric space; contraction; partial order.

1. Introduction and Preliminaries

In order to generalize the well-known Banach contraction principle, Nadler [15] introduced the Banach contraction principle for multivalued mappings in complete metric spaces. It is known that the theorem by Nadler has been extended and generalized in various directions by several authors, see [1, 2, 3, 9, 10] and the references therein. On the other hand, modular metric spaces are a natural and interesting generalization of classical modulars over linear spaces such as Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and others. The concept of modular metric spaces was introduced in [6, 7]. Here, we look at the modular metric space as the nonlinear version of the classical one introduced by Nakano [16] on the vector space and the modular function space introduced by Musielak [14] and Orlicz [17].

Recently, many authors studied the behavior of the electrorheological fluids, sometimes referred to as "smart fluids" (e.g., lithium polymetachrylate). A perfect model for these fluids is obtained by using Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, in the case p is a function [8]. In this paper, we generalize the notion of $O-$sets and then establish some fixed point theorems for $\perp - \alpha - \psi-$contraction

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multifunction in the setting of orthogonal modular metric spaces. As consequences of these results, we deduce some theorems in orthogonal modular metric spaces endowed with a graph and partial order. In the end, we establish some theorems for integral type contraction multifunctions and give some examples to demonstrate the validity of the results.

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function. For reasons of simplicity we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. [6, 7] A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If in the above definition we utilize the condition

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i) then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again ω is called convex if for $\lambda, \mu > 0$ and $x, y, z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu(z, y).$$

Remark 1.1. Note that if ω is a pseudomodular metric on a set X then the function $\lambda \to \omega_\lambda(x, y)$ is decreasing on $(0, +\infty)$ for all $x, y \in X$. That is, if $0 < \mu < \lambda$ then

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Definition 1.2. [6, 7] Suppose that ω be a pseudomodular on X and $x_0 \in X$ and fixed. So the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \quad \text{such that} \quad \omega_\lambda(x, x_0) < +\infty\}.$$

X_ω and X^*_ω are called modular spaces (around x_0).

It is evident that $X_\omega \subset X^*_\omega$ but this inclusion may be proper in general. Assume that ω be a modular on X, from [6, 7] we derive that the modular space X_ω can be equipped with a (nontrivial) metric induced by ω and given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\} \quad \text{for all} \quad x, y \in X_\omega.$$

Note that if ω is a convex modular on X then according to [6, 7] the two modular spaces coincide, i.e., $X^*_\omega = X_\omega$, and this common set can be endowed with the metric d^*_ω given by

$$d^*_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\} \quad \text{for all} \quad x, y \in X_\omega.$$

Such distances are called Luxemburg distances.

Example 2.1 presented by Abdou and Khamsi [1] is an important motivation for developing the modular metric spaces theory. Other examples may be found in [6, 7].

Definition 1.3. [13] Assume X_ω is a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Therefore,

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if and only if $\omega_\lambda(x_n, x) \to 0$, as $n \to +\infty$ for all $\lambda > 0$. x will be called the ω-limit of (x_n).

(2) $(x_n)_{n\in N}$ is called ω-Cauchy if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to +\infty$ for all $\lambda > 0$.

(3) M is called ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.

(4) M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.

(5) M is called ω-bounded if for all $\lambda > 0$ we have $\delta_\omega(M) = \sup\{\omega_\lambda(x, y); x, y \in M\} < +\infty$.

Definition 1.4. [6, 7] ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\} \subseteq X_\omega$ with $\lim_{n\to\infty} \omega_1(x_n, x) = 0$, we have

$$\omega_1(x, y) \leq \liminf_{n\to\infty} \omega_1(x_n, y)$$

for all $y \in X_\omega$.

But here we utilize the following version of the Fatou property.

Definition 1.5. ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\} \subseteq X_\omega$, ω-convergent to x, we get

$$\omega_\lambda(x, y) \leq \liminf_{n\to\infty} \omega_\lambda(x_n, y)$$

for all $y \in X_\omega$ and $\lambda > 0$.

Also we say ω satisfies the Δ_2-condition (see [2]), if $\lim_{n\to\infty}\omega(x_n,x)=0$ for some $\lambda>0$ implies $\lim_{n\to\infty}\omega(x_n,x)=0$ for all $\lambda>0$.

Definition 1.6. [5] Let M be a subset of the modular metric space X_ω.

- $CB(M)=\{C:\ C$ is nonempty ω-closed and ω-bounded subset of $M\}$
- $K(M)=\{C:\ C$ is nonempty ω-compact subset of $M\}$
- A Hausdorff modular metric $\Omega_\lambda(A,B)$ is defined on $CB(M)$ by

$$\Omega_\lambda(A,B)=\max\{\sup_{x\in A}\omega_\lambda(x,B),\sup_{y\in B}\omega_\lambda(A,y)\}$$

  where $\omega_\lambda(x,B)=\inf_{y\in B}\omega_\lambda(x,y)$.

Furthermore, let $T:M\to CB(M)$ be a multifunction. We say $x\in M$ is fixed point of T whence $x\in Tx$. We denote all fixed points of T by $Fix(T)$.

Lemma 1.1. *[5] Suppose that $A,B\in CB(X_\omega)$ and $a\in A$. Thus for $\epsilon>0$, there exists $b_\epsilon\in B$ such that*

$$\omega_\lambda(a,b_\epsilon)\leq\Omega_\lambda(A,B)+\epsilon$$

for all $\lambda>0$.

Asl et al. [3] defined the notion of α_*-admissible multifunction as follows.

Definition 1.7. Let $T:X\to 2^X$ and $\alpha:X\times X\to\mathbb{R}_+$. We say that T is α_*-admissible mapping if

$$\alpha(x,y)\geq 1 \quad \text{implies} \quad \alpha_*(Tx,Ty)\geq 1, \quad x,y\in X$$

where

$$\alpha_*(A,B)=\inf_{x\in A,\ y\in B}\alpha(x,y).$$

Denote Ψ the family of strictly increasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\Sigma_{n=1}^{\infty}\psi^n(t)<\infty$ for all $t>0$.

Eshaghi et al. [9] introduced the notion of orthogonal set and gave a real generalization of Banach's fixed point theorem in orthogonal metric spaces (For more details on orthogonal set, also see [4]).

Definition 1.8. [9] Let $X\neq\emptyset$ and $\perp\in X\times X$ be a binary relation. Assume that there exists $x_0\in X$ such that $x_0\perp x$ or $x\perp x_0$ for all $x\in X$. Then we say that X is an orthogonal set (briefly O-set). We denote the orthogonal set by $(X,\perp)$. Also suppose that $(X,\perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if $(\forall n;x_n\perp x_{n+1})$ or $(\forall n;x_{n+1}\perp x_n)$.

Definition 1.9. [9] We say a metric space X is an orthogonal metric space if $(X, \perp)$ is an O-set. Also $T : X \to X$ is $\perp-$continuous in $x \in X$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X if $\lim_{n\to\infty} d(x_n, x) = 0$, then $\lim_{n\to\infty} d(Tx_n, Tx) = 0$. Furthermore T is $\perp$-continuous if T is $\perp$-continuous in each $x \in X$. Also we say T is $\perp$-preserving if $Tx\perp Ty$ whence $x\perp y$. Finally X is orthogonally complete (in brief O-complete) if every Cauchy O-sequence is convergent.

Now we generalize the concept of $O-$set and introduce the notion of $O^\star-$modular metric space in the following ways.

Definition 1.10. Let $X \neq \emptyset$ and $\perp \in X \times X$ be a binary relation.

- Assume that there exists $x_0 \in X$ such that $x_0 \perp x$ for all $x \in X \setminus \{x_0\}$. Then we say that X is an orthogonal star set(briefly $O^\star$-set). We denote $O^\star$-set by $(X, \perp)$.
- We say x_0 is center of X and we denote the set of all centers of X by $\mathcal{C}(X)$.
- Also suppose that $(X, \perp)$ be an $O^\star$-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $O^\star$-sequence if $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 1.11. Let X_ω be a modular metric space and $M \subseteq X_\omega$.

- M is an $O^\star-$modular metric space if $(M, \perp)$ is an $O^\star$-set.
- $T : M \to M$ is $\perp^\star-$continuous in $x \in M$ if for each $O^\star$-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$, implies $\lim_{n\to\infty} \omega_\lambda(Tx_n, Tx) = 0$ for all $\lambda > 0$. Furthermore T is $\perp^\star$-continuous when T is $\perp^\star$-continuous in each $x \in M$.
- $T : M \to CB(M)$ is $\perp^{\star\star}-$continuous in $x \in M$ if for each $O^\star$-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$, implies $\lim_{n\to\infty} \Omega_\lambda(Tx_n, Tx) = 0$ for all $\lambda > 0$. Also T is $\perp^{\star\star}$-continuous when T is $\perp^{\star\star}$-continuous in each $x \in M$.
- $T : M \to M$ is $\perp^\star$-preserving if $Tx\perp Ty$ whence $x\perp y$.
- $T : M \to CB(M)$ is $\perp^{\star\star}$-preserving, when $x\perp y$ implies $u\perp v$ for all $u \in Tx$ and $v \in Ty$.
- Finally X_ω is $\omega - O^\star$-complete if every ω-Cauchy $O^\star$-sequence is convergent.

If $x_0 \perp y$ for all $y \in X$ then evidently $x_0 \perp y$ for all $y \in X \setminus \{x_0\}$. That is every O-set $(X, \perp)$ is an $O^\star-$set, but the converse is not true. The following simple example shows this fact.

Example 1.1. Let $X = [0, \infty)$. For $x, y \in X$, assume $x\perp y$ if $x < y$. Then by putting $x_0 = 0$, X is an $O^\star-$set. In fact $x_0 = 0 < x$ for all $x \in [0, \infty) \setminus \{x_0 = 0\}$. But $0 \not< 0$. That is $(X, \perp)$ is not O-set.

2. Main Results

To demonstrate our main theorems we need the following lemmas.

Lemma 2.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let B be an $\omega-$closed subset of X_ω. Then $x \notin B$ if and only if $\omega_\lambda(x, B) > 0$ for all $\lambda > 0$.*

Proof. Let $\omega_\lambda(x, B) > 0$ for all $\lambda > 0$. Now if $x \in B$ then $\omega_\lambda(x, B) = \inf_{y \in B} \omega_\lambda(x, y) = 0$ for all $\lambda > 0$, which is a contradiction. Hence $x \notin B$.

Let $x \notin B$. Now assume there exists $\lambda_0 > 0$ such that $\omega_{\lambda_0}(x, B) = \inf_{y \in B} \omega_{\lambda_0}(x, y) = 0$. Then there exists a sequence $\{y_n\}_{n \geq 0} \subseteq B$ such that $\lim_{n\to\infty} \omega_{\lambda_0}(x, y_n) = 0$. Δ_2-condition implies $\lim_{n\to\infty} \omega_\lambda(x, y_n) = 0$ for all $\lambda > 0$. That is $y_n \to z$ as $n \to \infty$. Now since B is $\omega-$closed, then $x \in B$, which is a contradiction. □

Lemma 2.2. *Let X_ω be a modular metric space such that ω satisfies the Fatou property. Let A, B be two subsets of X_ω where B is $\omega-$compact. Then for each $x \in A$ there exists $y \in B$ such that $\omega_\lambda(x, y) \leq \Omega_\lambda(A, B)$ for all $\lambda > 0$.*

Proof. Let $x \in A$. Then by using lemma 1.1 we can say for each $n \geq 1$ there exists $y_n \in B$ such that

$$\omega_\lambda(x, y_n) \leq \Omega_\lambda(A, B) + \frac{1}{n}.$$

On the other hand B is $\omega-$compact. Thus we may assume that $\{y_n\}$ $\omega-$converges to $y \in B$. Since ω satisfies the Fatou property, we get

$$\omega_\lambda(x, y) \leq \liminf_{n\to\infty} \omega_\lambda(x, y_n) \leq \Omega_\lambda(A, B),$$

for all $\lambda > 0$. □

Lemma 2.3. *Let X_ω be a modular metric space and $\emptyset \neq M \subseteq X_\omega$. Let $A, B \in CB(M)$ and $q > 1$. Then for each $x \in A$ there exists $y \in B$ such that $\omega_\lambda(x, y) < q\Omega_\lambda(A, B)$ for all $\lambda > 0$.*

Proof. If in lemma 1.1 we take $\epsilon = \frac{1}{2}(q-1)\Omega_\lambda(A, B)$ then for each $x \in A$ there exists $y \in B$ such that

$$\begin{aligned}\omega_\lambda(x, y) \leq \Omega_\lambda(A, B) + \epsilon \quad &= \Omega_\lambda(A, B) + \frac{1}{2}(q-1)\Omega_\lambda(A, B) < \Omega_\lambda(A, B) + (q-1)\Omega_\lambda(A, B)\\ &= q\Omega_\lambda(A, B).\end{aligned}$$

□

Now we are ready to prove our first theorem.

Theorem 2.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega-O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x\perp y \\ \alpha(x,y) \geq 1 \end{cases} \Longrightarrow \Omega_\lambda(Tx,Ty) \leq \psi\big(\omega_\lambda(x,y)\big). \tag{2.1}$$

Also suppose that the following assertion holds:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$,

(ii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Proof. From (i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. We know that $x_0 \in \mathcal{C}(M)$, that is $x_0 \perp y$ for all $y \in M \setminus \{x_0\}$. If $x_0 = x_1$ then x_0 is a fixed point of T. Hence we assume that $x_0 \neq x_1$. So $x_0 \perp x_1$. Therefore from (2.1) we have

$$\Omega_\lambda(Tx_0,Tx_1) \leq \psi(\omega_\lambda(x_0,x_1)). \tag{2.2}$$

Also if $x_1 \in Tx_1$ then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$. Then by using lemma 2.1 we have

$$0 < \omega_\lambda(x_1,Tx_1) \text{ for all } \lambda > 0. \tag{2.3}$$

Now if $q > 1$ then from lemma 2.3 there exists $x_2 \in Tx_1$ such that

$$\omega_\lambda(x_1,x_2) < q\Omega_\lambda(Tx_0,Tx_1) \text{ for all } \lambda > 0. \tag{2.4}$$

Since $\omega_\lambda(x_1,Tx_1) \leq \omega_\lambda(x_1,x_2)$, for all $\lambda > 0$ then from (2.3) and (2.4) we obtain

$$0 < \omega_\lambda(x_1,Tx_1) \leq \omega_\lambda(x_1,x_2) < q\Omega_\lambda(Tx_0,Tx_1) \text{ for all } \lambda > 0.$$

And so by (2.2) we get

$$0 < \omega_\lambda(x_1,Tx_1) \leq \omega_\lambda(x_1,x_2) < q\Omega_\lambda(Tx_0,Tx_1) \leq q\psi(\omega_\lambda(x_0,x_1)).$$

That is

$$0 < \omega_\lambda(x_1,x_2) < q\psi(\omega_\lambda(x_0,x_1)). \tag{2.5}$$

Note that $x_1 \neq x_2$ (since $x_1 \notin Tx_1$). Also since T is an α_*-admissible then $\alpha_*(Tx_0,Tx_1) \geq 1$. This implies

$$\alpha(x_1,x_2) \geq \alpha_*(Tx_0,Tx_1) \geq 1.$$

Further since T is an $\perp^{\star\star}-$preserving then $x_0 \perp x_1$ implies $u \perp v$ for all $u \in Tx_0$ and $v \in Tx_1$. This implies $x_1 \perp x_2$.

Therefore from (2.1) we have

$$\Omega_\lambda(Tx_1, Tx_2) \leq \psi(\omega_\lambda(x_1, x_2)). \tag{2.6}$$

Put $t_0 = \omega_\lambda(x_0, x_1)$. We know that $x_0 \neq x_1$. Let $B = \{x_1\}$. Then lemma 2.1 implies that $\omega_\lambda(x_0, x_1) > 0$ for all $\lambda > 0$. That is $t_0 > 0$. So from (2.5) we have $\omega_\lambda(x_1, x_2) < q\psi(t_0)$ where $t_0 > 0$. Now since ψ is strictly increasing then $\psi(\omega_\lambda(x_1, x_2)) < \psi(q\psi(t_0))$. Put

$$q_1 = \frac{\psi(q\psi(t_0))}{\psi(\omega_\lambda(x_1, x_2))}$$

and so $q_1 > 1$. If $x_2 \in Tx_2$ then x_2 is a fixed point of T. Hence we suppose that $x_2 \notin Tx_2$. Then

$$0 < \omega_\lambda(x_2, Tx_2) \text{ for all } \lambda > 0.$$

So there exists $x_3 \in Tx_2$ such that

$$0 < \omega_\lambda(x_2, x_3) < q_1\Omega_\lambda(Tx_1, Tx_2)$$

and then from (2.6) we get

$$0 < \omega_\lambda(x_2, x_3) < q_1\Omega_\lambda(Tx_1, Tx_2) \leq q_1\psi(\omega_\lambda(x_1, x_2)) = \psi(q\psi(t_0)).$$

Again since ψ is strictly increasing, then $\psi(\omega_\lambda(x_2, x_3)) < \psi(\psi(q\psi(t_0)))$. Put

$$q_2 = \frac{\psi(\psi(q\psi(t_0)))}{\psi(\omega_\lambda(x_2, x_3))}.$$

So $q_2 > 1$. If $x_3 \in Tx_3$ then x_3 is a fixed point of T. Hence we assume $x_3 \notin Tx_3$. Then

$$0 < \omega_\lambda(x_3, Tx_3) \text{ for all } \lambda > 0,$$

and so there exists $x_4 \in Tx_3$ such that

$$0 < \omega_\lambda(x_3, x_4) < q_2\Omega_\lambda(Tx_2, Tx_3). \tag{2.7}$$

Clearly $x_2 \neq x_3$. Also again since T is α_*-admissible and $\perp-$preserving then

$$\alpha(x_2, x_3) \geq 1 \text{ and } x_2 \perp x_3.$$

Then from (2.1) we have

$$\Omega_\lambda(Tx_2, Tx_3) \leq \psi(\omega_\lambda(x_2, x_3)),$$

and so from (2.7) we deduce that

$$\omega_\lambda(x_3, x_4) < q_2\Omega_\lambda(Tx_2, Tx_3) \leq q_2\psi(\omega_\lambda(x_2, x_3)) = \psi(\psi(q\psi(t_0))).$$

By continuing this process we obtain a sequence $\{x_n\}$ in X_ω such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$, $x_{n-1} \perp x_n$, $\alpha(x_{n-1}, x_n) \geq 1$ and $\omega_1(x_n, x_{n+1}) \leq \psi^{n-1}(q\psi(t_0))$ for all $n \in \mathbb{N}$. Let p be a given positive integer. Now we can write

$$\omega_\lambda(x_n, x_{n+p}) = \omega_{p\frac{\lambda}{p}}(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} \omega_{\frac{\lambda}{p}}(x_k, x_{k+1}) \leq \sum_{k=n}^{n+p-1} \psi^{k-1}(q\psi(t_0)).$$

Therefore $\{x_n\}$ is an $\omega-$Cauchy sequence. Since X_ω is an $\omega-$complete modular metric space then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is $\perp^{\star\star}-$continuous then

$$\lim_{n\to\infty} \Omega_\lambda(Tx_{n-1}, Tz) = 0$$

for all $\lambda > 0$. Let $q > 1$. From lemma 2.3 for each $x_n \in Tx_{n-1}$ there exist $y_n \in Tz$ such that

$$\omega_\lambda(x_n, y_n) < q\Omega_\lambda(Tx_{n-1}, Tz)$$

for all $\lambda > 0$. Then $\lim_{n\to\infty} \omega_\lambda(x_n, y_n) = 0$ for all $\lambda > 0$. Therefore

$$\omega_\lambda(z, y_n) \leq \omega_{\frac{\lambda}{2}}(z, x_n) + \omega_{\frac{\lambda}{2}}(x_n, y_n).$$

By taking limit as $n \to \infty$ in the above inequality we get $\omega_\lambda(z, y_n) = 0$, for all $\lambda > 0$. That is the sequence $\{y_n\}$ $\omega-$converges to z. Since Tz is $\omega-$closed then $z \in Tz$. $\square$

For multifunction T that is not $\perp^{\star\star}-$continuous we prove the following theorem.

Theorem 2.2. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ \alpha(x, y) \geq 1 \end{cases} \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big). \tag{2.8}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$\alpha(x_n, x) \geq 1 \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. As in the proof of theorem 2.1 we deduce an $O^\star$-sequence $\{x_n\}$ starting at x_0 is $\omega-$Cauchy and so $\omega-$converges to a point $z \in X_\omega$. Then from (ii) we have

$$\alpha(x_n, z) \geq 1 \text{ and } x_n \perp z.$$

So from (2.9) we have

$$\Omega_\lambda(Tx_{n-1}, Tz) \leq \psi(\omega_\lambda(x_{n-1}, z))$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequalities we get

$$\lim_{n\to\infty} \Omega_\lambda(Tx_{n-1}, Tz) = 0.$$

Now as in the proof of theorem 2.1 we get $z \in Tz$. $\square$

Example 2.1. let $X = \{1, 2, 3\}$ and define modular metric ω on X be defined by

$$\omega_\lambda(x, y) = \omega_\lambda(y, x) = \begin{cases} 0 & x = y, \\ \dfrac{1}{4\lambda} & x, y \in X \setminus \{2\}, \\ \dfrac{1}{2\lambda} & x, y \in X \setminus \{3\}, \\ \dfrac{5}{8\lambda} & x, y \in X \setminus \{1\}. \end{cases}$$

Suppose $T2 = \{1\}$ and $Tx = \{3\}$ for $x \neq 2$, $\alpha(x, y) = 1$ and $x \perp y$ if and only if $x < y$. Let $\psi(t) = \frac{t}{2}$. For $x \perp y$, we consider to the following cases:

- Let $x = 1$ and $y = 2$, then,

$$\Omega_\lambda(T1, T2) = \omega_\lambda(1, 3) = \frac{1}{4\lambda} = \psi(\omega_\lambda(1, 2)).$$

- Let $x = 1$ and $y = 3$, then,

$$\Omega_\lambda(T1, T3) = \omega_\lambda(3, 3) = 0 \leq \psi(\omega_\lambda(1, 3)).$$

- Let $x = 2$ and $y = 3$, then,

$$\Omega_\lambda(T2, T3) = \omega_\lambda(1, 3) = \frac{1}{4} \leq \frac{5}{16} = \psi(\omega_\lambda(2, 3)).$$

Therefore all conditions of theorem 2.2 holds and T has a fixed point.

Example 2.2. Let $X = \mathbb{R}$, $M = [0, \infty)$ and $\omega_\lambda(x, y) = \frac{1}{\lambda}|x - y|$. Define $T : M \longrightarrow CB(M)$ by

$$Tx = \begin{cases} [\dfrac{x}{4}, \dfrac{x}{2}] & 0 \leq x \leq 1 \\ [\frac{e^{-x}}{2}, e^{-x}] & x > 0 \end{cases}$$

and $\alpha : M \times M \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3 & x, y \in [0, 1], \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that T is an α_*-admissible. Let $\psi(t) = \frac{7t}{8}$ for all $t \geq 0$ and $x \perp y$ if $x \leq y$. Let $x \perp y$ and $\alpha(x, y) \geq 1$. Then $x, y \in [0, 1]$ and $0 \leq x \leq y \leq 1$. Then we write

$$\Omega_\lambda([\frac{x}{4}, \frac{x}{2}], [\frac{y}{4}, \frac{y}{2}]) = \frac{1}{2\lambda}\omega_\lambda(x, y) \leq \frac{7}{8\lambda}\omega_\lambda(x, y) = \psi(\omega_\lambda(x, y)).$$

If $\{x_n\} \subset X$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to +\infty$, then $x \in [0, 1]$ and $x_n \leq x$ for all $n \geq 0$. That is $\alpha(x_n, x) \geq 1$ and $x_n \perp x$. Hence all conditions of theorem 2.2 holds and T has a fixed point. Let $x = 0$ and $y = 1$. So for usual metric $d(x, y) = |x - y|$ we have

$$\alpha(0, 1)H(T0, T1) = \frac{3}{2} > 1 = d(0, 1) > \psi(d(0, 1)).$$

Therefore theorem 2.1 of [3] can not be applied for this example.

If in theorem we take $\alpha(x, y) = 1$, then we obtain the following corollary.

Corollary 2.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$x \perp y \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big).$$

Then T has a fixed point.

Corollary 2.2. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible multifunction. Assume that for $\psi \in \Psi$,*

$$\alpha(x, y) \geq 1 \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big). \tag{2.9}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) if $\{x_n\}$ be a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in M$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define a binary relation $\perp \in M \times M$ by $x \perp y$ if $(x, y) \in M \times M$. Then $x \perp y$ for all $x, y \in M$. That is $(M, \perp)$ is an $O^\star-$set and $\mathcal{C}(M) = M$. Clearly $(x, y) \in M \times M$ and $(u, v) \in M \times M$ for all $x, y \in M$ and all $u \in Tx$ and $v \in Ty$.

That is $x \perp y$ and $u \perp v$ for all $x, y \in M$ and all $u \in Tx$ and $v \in Ty$. Then T is a $\perp^{\star\star}-$preserving multifunction. From (i) there exist $x_0 \in M = \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that $\{x_n\}$ be an $O^{\star}-$sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in M$ as $n \to \infty$. Thus from (ii) we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Also clearly $(x_n, x) \in M \times M$ for all $n \in \mathbb{N}$. Now if $x \perp y$ and $\alpha(x, y) \geq 1$ then $(x, y) \in M \times M$ and $\alpha(x, y) \geq 1$ and so from (2.9) we get $\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big)$. Hence all conditions of theorem 2.2 hold and T has a fixed point. □

If in corollary we take $\alpha(x, y) = 1$ then we obtain the following result.

Corollary 2.3. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be a multifunction. Assume that for $\psi \in \Psi$,*

$$\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big).$$

holds for all $x, y \in M$. Then T has a fixed point.

If in the above corollary we take $\psi(t) = rt$ where $r \in [0, 1)$ then we deduce the following result.

Corollary 2.4. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be a multifunction. Assume that for $r \in [0, 1)$,*

$$\Omega_\lambda(Tx, Ty) \leq r\omega_\lambda(x, y).$$

holds for all $x, y \in M$. Then T has a fixed point.

The following corollary is Theorem 2.1 of Asl et al. [3] in the setting of modular metric spaces.

Corollary 2.5. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible multifunction. Assume that for $\psi \in \Psi$*

$$\alpha(x, y)\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big) \tag{2.10}$$

holds for all $x, y \in M$. Also suppose that the following assertions hold:

(i) there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) if $\{x_n\}$ be a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in M$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Let $\alpha(x, y) \geq 1$. Then from (2.10) we get $\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big)$. Hence all conditions of corollary 2. hold and T has a fixed point. □

3. Some Results in Modular Metric spaces endowed with a graph

As in [11], let (X_ω, ω) be a modular metric space and Δ denotes the diagonal of the cartesian product of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover we may treat G as a weighted graph (see [12], p. 309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x,\ x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$.

Definition 3.1. [11] Let (X, d) be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is

$$\text{for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$$

and T decreases the weights of the edges of G in the following way:

$$\exists\, \alpha \in (0, 1) \text{ such that for all } x, y \in X, \quad (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leqslant \alpha d(x, y).$$

Definition 3.2. [11] A mapping $T : X \to X$ is called G-continuous if given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \,\text{as}\, n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \,\text{for all}\, n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

In this section we assert some $\perp - \psi-$contraction multifunction type fixed point results in O^*-modular metric spaces endowed with a graph G which can be deduced easily from our presented theorems.

Theorem 3.1. *Let X_ω be a modular metric space endowed with a graph G such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be a $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ (x, y) \in E(G) \end{cases} \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big). \tag{3.1}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$,

(ii) if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$,

(iii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Proof. Define $\alpha : X_\omega \times X_\omega \to [0,+\infty)$ by $\alpha(x,y) = \begin{cases} 2, & \text{if } (x,y) \in E(G) \\ 0, & \text{otherwise} \end{cases}$.
First we show that T is an α_*-admissible multifunction. Let $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. From (ii) we have $(u,v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$. Then $\alpha_*(Tx,Ty) = \inf\{\alpha(u,v) : \ u \in Tx,\ v \in Ty\} = 2 \geq 1$. Thus T is an α_*-admissible multifunction. From (i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$. That is $\alpha(x_0,x_1) \geq 1$. Assume that $x \perp y$ and $\alpha(x,y) \geq 1$. Thus $x \perp y$ and $(x,y) \in E(G)$. Hence from (4.1) we have $\Omega_\lambda(Tx,Ty) \leq \psi\big(\omega_\lambda(x,y)\big)$. Therefore all conditions of theorem 2.1 hold and T has a fixed point. □

Theorem 3.2. *Let X_ω be a modular metric space endowed with a graph G such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ (x,y) \in E(G) \end{cases} \quad \Longrightarrow \Omega_\lambda(Tx,Ty) \leq \psi\big(\omega_\lambda(x,y)\big).$$

Also suppose that the following assertions hold:

(i) *there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$,*

(ii) *if $(x,y) \in E(G)$, then $(u,v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$,*

(iii) *if $\{x_n\}$ be an $O-$sequence in X_ω such that $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then*

$$(x_n,x) \in E(G) \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X_\omega \times X_\omega \to [0,+\infty)$ as in the proof of theorem 3.1. Let $\{x_n\}$ be a $O^\star-$sequence in M such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. From (iii) we get $(x_n,x) \in E(G)$ and $x_n \perp x$. That is $\alpha(x_n,x) \geq 1$ and $x_n \perp x$ for all $n \in \mathbb{N} \cup \{0\}$. Similar to the proof of theorem 3.1 we can prove that other conditions of theorem 2.2 are satisfied. Therefore all conditions of theorem 2.2 hold and T has a fixed point. □

4. Some Results in Modular Metric spaces endowed with a partial order

The existence of fixed points in partially ordered sets has been considered in [18]. Let X_ω be a nonempty set. If X_ω be a modular metric space and $(X_\omega, \preceq)$ be a partially ordered set then X_ω is called a partially ordered modular metric space. Two elements $x, y \in X_\omega$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

In this section we will show that some $\perp - \psi-$contraction multifunction type fixed point results in O^*-modular metric spaces endowed with a partial order $\preceq$ can be deduced easily from our presented theorems.

Theorem 4.1. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ x \preceq y \end{cases} \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big). \tag{4.1}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Proof. Define $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ by $\alpha(x, y) = \begin{cases} 2, & \text{if } x \preceq y \\ 0, & \text{otherwise} \end{cases}$. Let $\alpha(x, y) \geq 1$ then $x \preceq y$. From (ii) we have $u \preceq v$ for all $u \in Tx$ and $v \in Ty$. Then $\alpha_*(Tx, Ty) = \inf\{\alpha(u, v) : \ u \in Tx, \ v \in Ty\} = 2 \geq 1$. Thus T is an α_*-admissible multifunction. From (i) there exists $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$. That is $\alpha(x_0, x_1) \geq 1$. Assume that $x \perp y$ and $\alpha(x, y) \geq 1$. Thus $x \perp y$ and $x \preceq y$. Hence from (4.1) we have $\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big)$. Therefore all conditions of Theorem 2.1 hold and T has a fixed point. $\square$

Theorem 4.2. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ x \preceq y \end{cases} \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$x_n \preceq x \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ as in the proof of theorem 3.1. Let $\{x_n\}$ be a $O^\star-$sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (iii) we get $x_n \preceq x$ and $x_n \perp x$. That is $\alpha(x_n, x) \geq 1$ and $x_n \perp x$ for all $n \in \mathbb{N} \cup \{0\}$. Similar to the proof of theorem 3.1 we can prove that other conditions of theorem 2.2 are satisfied. Therefore all conditions of Theorem 2.2 hold and T has a fixed point. $\square$

5. Some Integral type contractions

Let Φ denote the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

- every $\phi \in \Phi$ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$,
- for any $\phi \in \Phi$ and any $\epsilon > 0$, $\int_0^\epsilon \phi(\tau)d\tau > 0$.

Following arguments similar to those in Theorem 2.1 and 2.2, we can prove the following theorems.

Theorem 5.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ \alpha(x, y) \geq 1 \end{cases} \implies \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertion holds:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Theorem 5.2. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ \alpha(x, y) \geq 1 \end{cases} \implies \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) *there exist* $x_0 \in \mathcal{C}(M)$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$,

(ii) *if* $\{x_n\}$ *be an* $O-$*sequence in* X_ω *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *with* $x_n \to x$ *as* $n \to \infty$, *then*

$$\alpha(x_n, x) \geq 1 \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T *has a fixed point.*

As consequences of the above theorems we can deduce the following results in the setting of $O^\star-$modular metric space endowed with a graph G or a partial order $\preceq$.

Theorem 5.3. *Let* X_ω *be a modular metric space endowed with graph* G *such that* ω *satisfies* Δ_2-*condition. Let* $(M, \perp)$ *be a nonempty* $\omega - O^*$*-complete subset of* X_ω. *Let* $T : M \to CB(M)$ *be an* $\perp^{\star\star}-$*preserving multifunction. Assume that for* $\psi \in \Psi$,

$$\begin{cases} x \perp y \\ (x, y) \in E(G) \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau) d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau) d\tau\Big).$$

Also suppose that the following assertions hold:

(i) *there exist* $x_0 \in \mathcal{C}(M)$ *and* $x_1 \in Tx_0$ *such that* $(x_0, x_1) \in E(G)$,

(ii) *if* $(x, y) \in E(G)$, *then* $(u, v) \in E(G)$ *for all* $u \in Tx$ *and* $v \in Ty$,

(iii) T *is* $\perp^{\star\star}-$*continuous.*

Then T *has a fixed point.*

Theorem 5.4. *Let* X_ω *be a modular metric space endowed with graph* G *such that* ω *satisfies* Δ_2-*condition. Let* $(M, \perp)$ *be a nonempty* $\omega - O^*$*-complete subset of* X_ω. *Let* $T : M \to CB(M)$ *be an* $\perp^{\star\star}-$*preserving multifunction. Assume that for* $\psi \in \Psi$,

$$\begin{cases} x \perp y \\ (x, y) \in E(G) \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau) d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau) d\tau\Big).$$

Also suppose that the following assertions hold:

(i) *there exist* $x_0 \in \mathcal{C}(M)$ *and* $x_1 \in Tx_0$ *such that* $(x_0, x_1) \in E(G)$,

(ii) *if* $(x, y) \in E(G)$, *then* $(u, v) \in E(G)$ *for all* $u \in Tx$ *and* $v \in Ty$,

(iii) *if* $\{x_n\}$ *be an* $O-$*sequence in* X_ω *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}$ *with* $x_n \to x$ *as* $n \to \infty$, *then*

$$(x_n, x) \in E(G) \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Theorem 5.5. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ x \preceq y \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Theorem 5.6. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ x \preceq y \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$x_n \preceq x \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

REFERENCES

1. Afrah A.N. Abdou and Mohamed A. Khamsi, On the fixed points of nonexpansive mappings in Modular Metric Spaces, Fixed Point Theory and Applications 2013, 2013:229.
2. Afrah A.N. Abdou and Mohamed A. Khamsi, Fixed points of multivalued contraction mappings in modular metric spaces, Fixed Point Theory and Applications 2014, 2014:249
3. J.H. Asl, SH. Rezapour, N. Shahzad, On fixed point of α-contractive multifunctions, Fixed Point Theory Appl., 2012, 2012:212.
4. H. Baghani, M. Eshaghi Gordji, M. Ramezani, Orthogonal sets: Their relation to the axiom choice and a generalized fixed point theorem, Journal of Fixed Point Theory and Applications, (2016). DOI 10.1007/s11784-016-0297-9
5. P. Chaipunya, C. Mongkolkeha, W. Sintunavarat, P. Kumam, Fixed-Point Theorems for Multivalued Mappings in Modular Metric Spaces, Abstract and Applied Analysis, vol. 2012, Article ID 503504, 14 pages. doi:10.1155/2012/503504
6. V.V. Chistyakov, *Modular metric spaces, I: Basic concepts,* Nonlinear Anal. **72**(1) (2010), 1-14.
7. V.V. Chistyakov, *Modular metric spaces, II: Application to superposition operators,* Nonlinear Anal. **72**(1) (2010), 15-30.
8. L. Diening, *Theoretical and numerical results for electrorheological fluids*, Ph.D. thesis (2002),University of Freiburg, Germany.
9. M. Eshaghi, M. Ramezani, M.D.L. Sen, Y.J. Cho, *On orthogonal sets and Banach's fxed point theorem*, accepted in Fixed point theorey.
10. N. Hussain, P. Salimi and A. Latif, Fixed point results for single and set-valued $\alpha - \eta - \psi$ -contractive mappings, Fixed Point Theory Appl., 2013, 2031:212
11. J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 1 (136) (2008) 13591373.
12. R. Johnsonbaugh, Discrete Mathematics, Prentice-Hall, Inc., New Jersey, 1997.
13. Ch. Mongkolkeha, W.Sintunavarat, P.Kumam, Fixed point theorems for contraction mappings in modular metric spaces, Fixed Point Theory and Applications 2011, 2011:93
14. J. Musielak, *Orlicz spaces and modular spaces,* Lecture Notes in Math., vol. **1034**, Springer, Berlin (1983).
15. S.B.Nadler, Jr., *Multi-valued contraction mappings*, Pacific Journal of Mathematics, vol. 30, pp. 475 488, 1969.
16. H. Nakano, *Modulared Semi-Ordered Linear Spaces,* Maruzen, Tokyo (1950).
17. W. Orlicz, *Collected Papers, Part I, II,* PWN Polish Scientific Publishers, Warsaw (1988).
18. A.C.M. Ran and M.C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132 (2004), 1435-1443.

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